

Non-monotonic orbital velocity profiles around rapidly rotating Kerr-(anti-)de Sitter black holes

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Abstract. It has been recently demonstrated that the orbital velocity profile around Kerr black holes in the equatorial plane as observed in the locally non-rotating frame exhibits a non-monotonic radial behaviour. We show here that this unexpected minimum–maximum feature of the orbital velocity remains if the Kerr vacuum is generalized to the Kerr-de Sitter or Kerr–anti–de Sitter metric. This is a new general relativity effect in Kerr spacetimes with non-vanishing cosmological constant. Assuming that the profile of the orbital velocity is known, this effect constrains the spacetime parameters.

PACS numbers: 04.70.-s, 04.20.Jb, 98.80.-k

Received 15 January 2007, in final form 5 April 2007
Accepted for publication in *Classical and Quantum Gravity*

1. Introduction

Rotating black holes are described by the Kerr solution of general relativity [1]. Matter particles can perform stable circular orbits in the equatorial plane around these compact objects as long as the orbital radius is greater than the marginally stable orbit [2]. The introduction of a locally non-rotating frame (LNRF) offers a fairly easy way for studying particle motions in the Kerr geometry [3]. In a sense, the observer's frame co-rotates with spacetime thereby cancelling frame-dragging effects as much as possible. Recently, an unexpected behaviour of the radial dependence of the orbital LNRF velocity of test masses with a Keplerian angular velocity distribution around a rapidly rotating, non-charged black hole has been discovered [4]: In vacuum spacetime of very fast spinning Kerr black holes with the rotational parameter $a > 0.9953$, there occurs an unexpected dip of the orbital LNRF velocity as function of orbital radius. This minimum-maximum structure of the orbital velocity emerges close to the black hole, at radii $r < 1.8$. Orbital radii are given in units of the gravitational radius that is defined as $r_g = GM/c^2$ with Newton's constant G , vacuum speed of light c and black hole mass M . We set $G=c=M=1$ throughout the paper for convenience. The gradient of the orbital velocity is positive in a small radial range. The radii associated with the local extrema are greater than the innermost stable circular orbit (ISCO) and smaller than the last stable orbit at $r = 2$ for any value of a . Hence, this effect occurs always within the ergosphere of a Kerr black hole. The velocity difference between the extrema, i.e. the slow down amounts approximately to 1% of c at Thorne's spin limit $a = 0.998$ [5] but is higher for larger black hole spin. The non-monotonic behaviour of the orbital LNRF velocity is a pure effect of general relativity and has been overlooked up to 2004. A follow-up investigation has shown that the non-monotonic behaviour remains even for non-Keplerian distributions of angular momentum ($l = \text{const}$) in the Kerr vacuum [6]. The critical value of the black hole spin that guarantees the emergence of the effect is higher in this case, i.e. $a > 0.99979$.

2. Kerr spacetimes with non-vanishing Λ

The motivation for this work is the question whether the non-monotonicity occurs also in more general Kerr spacetimes or might actually been removed. In the present work, the LNRF orbital velocity is generalized for Kerr-de Sitter and Kerr-anti-de Sitter spacetimes, i.e. including the cosmological constant Λ . As for the Kerr spacetime proper, we use the Boyer-Lindquist form [7] to study the motion of a particle. Then, the line element of the ordinary Kerr geometry holds

$$ds^2 = -\alpha^2 dt^2 + \tilde{\omega}^2 (d\phi - \omega dt)^2 + \rho^2/\Delta dr^2 + \rho^2 d\theta^2, \quad (1)$$

with the functions ($G = c = M = 1$)

$$\alpha = \rho\sqrt{\Delta}/\Sigma, \quad (2)$$

$$\Delta = r^2 - 2r + a^2, \quad (3)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad (4)$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad (5)$$

$$\omega = 2ar/\Sigma^2, \quad (6)$$

$$\tilde{\omega} = \Sigma \sin \theta / \rho, \quad (7)$$

where M and a denote black hole mass and spin, respectively.

The line element for the Kerr-de Sitter (KdS, $\Lambda > 0$) or Kerr-anti-de Sitter (KadS, $\Lambda < 0$) metric, respectively, is significantly more complicated [8, 9, 10]

$$ds^2 = -\frac{\Delta_r}{\chi^2 \rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Delta_\theta \sin^2 \theta}{\chi^2 \rho^2} \times [a dt - (r^2 + a^2) d\phi]^2 + \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right), \quad (8)$$

where we have the generalized functions

$$\Delta_r = (r^2 + a^2) (1 - \frac{1}{3} \Lambda r^2) - 2r, \quad (9)$$

$$\Delta_\theta = 1 + \frac{1}{3} \Lambda a^2 \cos^2 \theta, \quad (10)$$

$$\chi = 1 + \frac{1}{3} \Lambda a^2. \quad (11)$$

Λ denotes the cosmological constant. These functions reduce to the ordinary Kerr geometry by setting $\Lambda = 0$:

$$\Delta_r = r^2 + a^2 - 2r \equiv \Delta, \quad (12)$$

$$\Delta_\theta = 1, \quad (13)$$

$$\chi = 1. \quad (14)$$

Now, we aim to cast the KdS/KadS line element into a form analogous to equation (1)

$$ds^2 = -\alpha_\Lambda^2 dt^2 + \tilde{\omega}_\Lambda^2 (d\phi - \omega_\Lambda dt)^2 + \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right), \quad (15)$$

where $\alpha_\Lambda, \tilde{\omega}_\Lambda, \omega_\Lambda$ are the generalizations of the functions $\alpha, \tilde{\omega}, \omega$, respectively. Indeed, this is possible and we find

$$\omega_\Lambda = \frac{a [(r^2 + a^2) \Delta_\theta - \Delta_r]}{\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta}, \quad (16)$$

$$\alpha_\Lambda^2 = \frac{\Delta_r \Delta_\theta \rho^2}{\chi^2 [\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta]}, \quad (17)$$

$$\tilde{\omega}_\Lambda^2 = \frac{\sin^2 \theta}{\chi^2 \rho^2} [\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta]. \quad (18)$$

Note that the first expression represents the generalization of the frame-dragging frequency, ω_Λ . Considering that the shift vector in the Kerr geometry satisfies $\beta^\phi = -\omega$ in Boyer-Lindquist form [11], it is interesting to investigate its generalization, $\beta_\Lambda^\phi = -\omega_\Lambda$. It is easy to show that ω_Λ is larger for increasing values of the cosmological constant, i.e. the frame-dragging effect becomes stronger with increasing Λ .

Further, it is even possible to reverse frame-dragging with a sufficiently low and negative Λ . In this case, the radial position where the reversion takes place is determined by the root of ω_Λ .

With these generalizations it is straightforward to compute the orbital velocity component relative to the LNRF for Kerr spacetimes with non-zero Λ just analogous to the ordinary Kerr geometry, see e.g. [12]:

$$v_\Lambda^{(\phi)} = \tilde{\omega}_\Lambda \frac{\Omega - \omega_\Lambda}{\alpha_\Lambda} = \frac{\sin \theta [\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta]}{\rho^2 \sqrt{\Delta_r \Delta_\theta}} \left\{ \Omega - \frac{a [(r^2 + a^2) \Delta_\theta - \Delta_r]}{\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta} \right\}. \quad (19)$$

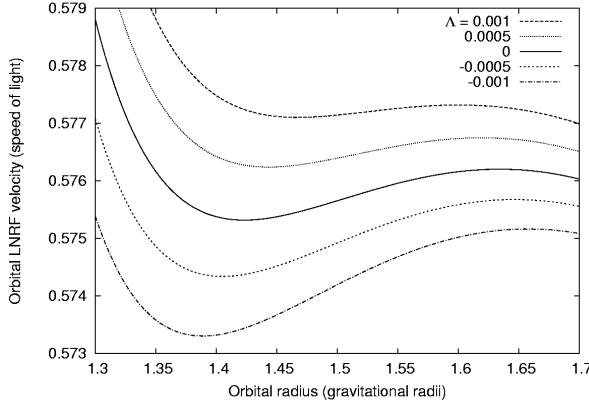


Figure 1. Orbital LNRF velocity profiles around a black hole with $a = 0.996$ for Keplerian orbits in the equatorial plane ($\theta = \pi/2$) and different values of the cosmological constant, $\Lambda = 10^{-3}, 5 \times 10^{-4}, 0, -5 \times 10^{-4}, -10^{-3}$. The orbital velocity is given in units of c ($G = c = M = 1$).

We investigate this completely general radial profile of the orbital LNRF velocity by specifying the angular velocity Ω for two cases: Keplerian or constant specific angular momentum rotation. Both distributions are motivated by astrophysical situations. Keplerian angular momentum distributions can be found in accretion flows described by the standard disc model [13]. Torus solutions exhibit constant angular momentum distributions [14].

2.1. Keplerian distribution of the specific angular momentum

The non-monotonic orbital velocity profile has originally been discovered for prograde orbits. A similar effect has not been found by us for retrograde orbits. Hence we assume a prograde (+) Keplerian angular velocity distribution of the orbiting test particles that satisfies

$$\Omega = \Omega_K^+ = \frac{1}{\sqrt{r^3 + a}}, \quad (20)$$

we can compute the velocity profile from equation (19). The result is shown in figure 1 which demonstrates that the minimum–maximum structure depends on Λ . As visible in the plot, a cosmological constant close to zero self-consistently approaches the ordinary case for the Kerr geometry. Further, the strength of the slow-down effect increases as Λ decreases, i.e. Kerr–anti–de Sitter ($\Lambda < 0$) exhibits a stronger effect than Kerr–de Sitter ($\Lambda > 0$).

2.2. Constant specific angular momentum distribution

Particles around black holes may not follow Keplerian orbits but show some distribution of the specific angular momentum l , e.g. $l = \text{const}$. In this case, the general expression for the angular velocity in the Kerr geometry can be specified for

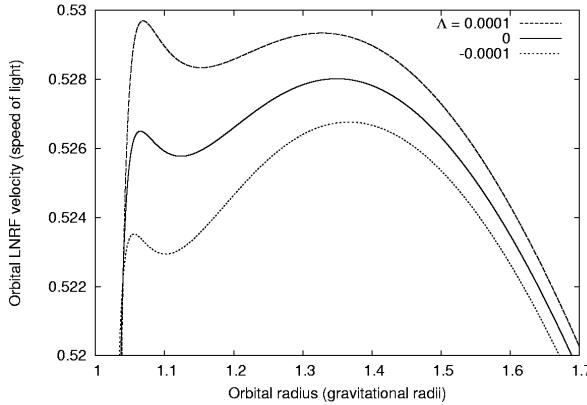


Figure 2. Orbital LNRF velocity profiles around a black hole with $a = 0.9999$ for $l = \text{const}$ orbits in the equatorial plane ($\theta = \pi/2$) and different values of the cosmological constant, $\Lambda = 10^{-4}, 0, -10^{-4}$.

the Kerr-(anti-)de Sitter geometry

$$\Omega = -\frac{lg_{tt} + g_{t\phi}}{lg_{t\phi} + g_{\phi\phi}} = \omega_\Lambda + \frac{\alpha_\Lambda^2}{\tilde{\omega}_\Lambda^2} \frac{l}{1 - \omega_\Lambda l}. \quad (21)$$

The specific angular momentum l has to be chosen in a range between marginally stable and marginally bound orbits, i.e. $l_{\text{ms}} \leq l \leq l_{\text{mb}}$, see [14]. Fixing l in this interval again reveals distinct minimum–maximum structures in the velocity profile. Figure 2 displays the velocity profile computed from equations (19) and (21) for a Kerr black hole with $a = 0.9999$ and $l = l(r = 1.0785) = l_{\text{ms}}$. In figure 2 the values for a and l are chosen as an example and to make our work comparable to the work of [6]. For $\Lambda = 0$ their result is confirmed (solid curve). The sign of Λ can be easily determined by comparing the profile’s position relative to the case $\Lambda = 0$. Similar to the Keplerian distribution, the trend remains: a negative Λ amplifies the slow–down effect.

3. An interplay of black hole spin and Λ

The examples discussed so far demonstrate an interesting interplay of black hole spin a and Λ . We have investigated this more deeply since the interaction seems to control the occurrence and modulation depth of the minimum–maximum structure. We consider only the equatorial plane ($\theta = \pi/2$) and compute the gradient of the orbital LNRF velocity for prograde Keplerian angular momentum distribution, $\Omega = \Omega_K^+$, and find

$$\begin{aligned} \frac{\partial v_\Lambda^{(\phi)}}{\partial r} = & [a^6 \Lambda r (9 - \Lambda r^2) + 2a^5 \Lambda r^{3/2} (\Lambda r^3 - 3) + a^4 (18 + 27r - 6\Lambda r^2 + 9\Lambda r^3 - 4\Lambda^2 r^5) \\ & + 2a^3 \sqrt{r} (-9 - 27r + 12\Lambda r^3 + 3\Lambda r^4 + \Lambda^2 r^6) \\ & - 3a^2 r^2 (30 - 12r + 6\Lambda r^2 + 3\Lambda r^3 + \Lambda^2 r^5) + 6a r^{5/2} (27 - 15r + 3\Lambda r^3 + \Lambda r^4) \\ & + 9r^5 (1 - \Lambda r^2)] / \{2 \sqrt{3r} (a + r^{3/2})^2 \sqrt{a^2 (3 - \Lambda r^2) - r (6 - 3r + \Lambda r^3)} \\ & \times [a^2 (\Lambda r^2 - 3) + r (6 - 3r + \Lambda r^3)]\}. \end{aligned} \quad (22)$$

Analogously, the gradient of the orbital LNRF velocity can be analytically computed by plugging equation (21) into equation (19), i.e. for constant angular momentum distributions. This gradient satisfies

$$\begin{aligned} \frac{\partial v_{\Lambda}^{(\phi)}}{\partial r} = & \{ l[-a^6\Lambda^2r^3 + a^5\Lambda^2r^3l - a^4(-18 + \Lambda^2r^5 + 3\Lambda r^2(5 + 2r)) \\ & + a^3l(-18 + \Lambda^2r^5 + 3\Lambda r^2(5 + r)) - 3a^2r(18 - 9r + 3r^2 + 3\Lambda r^3 + 2\Lambda r^4) \\ & + 3alr(18 - 12r + 3\Lambda r^3 + \Lambda r^4) - 9r^4(r - 3)]\} / \\ & \{ \sqrt{a^2 - 2r + r^2 - a^2\Lambda r^2/3 - \Lambda r^4/3} \\ & \times [a^4\Lambda r - a^3\Lambda lr + 3r^3 - al(6 + \Lambda r^3) + a^2(6 + 3r + \Lambda r^3)]^2 \}. \end{aligned} \quad (23)$$

We study both gradient equations in the following and start with the prograde Keplerian angular velocity. The roots of the gradient determine the radial positions of the local extrema. In general, the existence and the positions of extrema are controlled by a and Λ . With decreasing Λ the slow-down effect is enhanced for fixed and sufficiently high black hole spin. A gradient with only one root has a point of inflection in the velocity profile, whereas more roots yield minima and maxima of the slow-down structure. For each given Λ , there exists a critical value of the black hole spin a_c that is associated with a gradient vanishing at one specific radius. For spin values greater than a_c , there exists a radial range where the gradient becomes positive. In the Kerr vacuum ($\Lambda = 0$), it is $a_c > 0.9953$ for Keplerian orbiters. However, generalizing to non-zero Λ it is even possible that a high value of Λ is associated with a lower value of critical spin. A graphical analysis of the gradient for fixed Λ delivers the critical spins a_c beyond which the minimum–maximum structure appears. The critical spin a_c scales linearly with Λ for $|a_c| < 1$ (figure 3). The study for ordinary Kerr black holes by Aschenbach [4] is confirmed and generalized to cases with non-zero cosmological constant.

Unexpectedly, a similar linear correlation between a_c and Λ can be found for $l = \text{const}$ angular momentum distributions (figure 4). Both the slope and the offset of a for $\Lambda = 0$ differ slightly. The critical spin grows more steeply and the offset is smaller for Keplerian angular momentum distributions. Further, the linear correlation is constrained to a narrow interval of the cosmological constant. We find in our analysis that the minimum–maximum structure appears only for $-0.001 \lesssim \Lambda \lesssim 0.00017$.

4. Discussion

The slow-down effect occurs for rapidly spinning black holes, $a \gtrsim 0.99$. Both observations and theory suggest the existence of rapidly spinning black holes. Rapid rotation of supermassive black holes is suggested by the observation of broad iron K X-ray lines [15], and by flares observed from the galactic centre black hole [16, 17, 18]. Black hole theory suggests that supermassive black holes are endowed with high angular momentum due to the black hole growth history [19, 20]. Jet launching mechanisms such as the Blandford–Znajek process [21] also involve rapidly spinning black holes. General relativistic magnetohydrodynamics simulations support this idea because strong outflows are driven by Poynting fluxes near black holes only for high spins [22, 23].

If the minimum–maximum structure around a fast spinning black hole could be detected, its radial profile constrains both black hole spin and Λ .

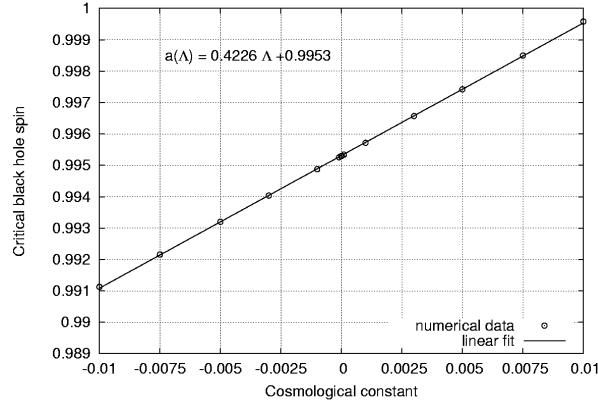


Figure 3. Critical spin values as function of Λ for Keplerian angular momentum distribution. The limiting values for the black hole spin a_c are plotted as function of Λ for Keplerian distributions of the specific angular momentum. There appears to be an almost positive linear correlation between a_c and Λ . For spins larger than a_c the slow-down effect occurs. However, the linear relation breaks down for $\Lambda > 0.011$, because of $a > 1$.

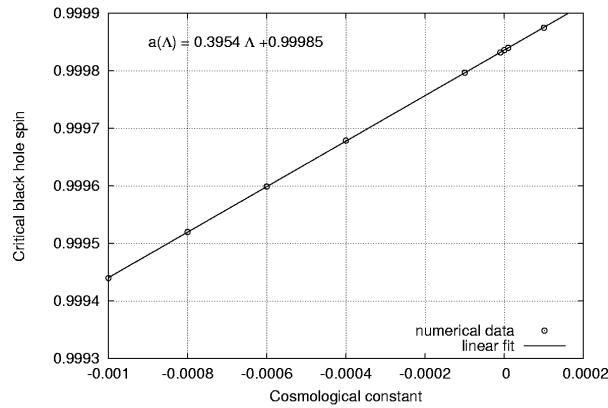


Figure 4. Critical spin values as function of Λ for $l = \text{const}$ angular momentum distribution. The limiting values for the black hole spin a_c are plotted as function of Λ for $l = \text{const}$ specific angular momentum distributions. Similar to the Keplerian case an positive linear correlation between a_c and Λ remains. The analysis of the orbital velocity profiles shows that the the minimum–maximum structure occurs only for $-0.001 \lesssim \Lambda \lesssim 0.00017$.

If black hole mass and spin were known from observations, e.g. from quasi-periodic oscillations, it might be possible to constrain the cosmological constant. But only extraordinarily high values of Λ would produce a significant change in the feature as demonstrated in the numerical examples. The cosmic concordance model [24] suggests that Λ is by many orders of magnitudes too small to enforce a significant change in the minimum–maximum structure. Currently, there is no hope to detect this for the Kerr black hole candidates. On the other hand, we would like to stress that theory of dark energy physics is limited by a number of uncertainties allowing a plethora of models. The extremes include proposals advocating for a cosmological constant that was significantly higher in earlier cosmological epochs, e.g. involving false vacua [25], or even that there is no need for a cosmological constant because density inhomogeneities drive a modification of cosmic expansion [26]. Another model involves a modification of 4D gravity [27]. The new GR effect outlined here might be exploited to probe general relativity for the strong field case. It is currently hard to say whether this new effect has any practical implication for cosmological studies.

5. Conclusions

We conclude that a very fast spinning black hole ($a > 0.991$ for $\Lambda < 0$ or $a > 0.9953$ for $\Lambda > 0$) causes a slow down of the orbital velocity at distances within two gravitational radii. Closer to the rotating black hole the orbital velocity steeply increases again – just as expected from Newtonian physics. It has been demonstrated in this paper that the ‘GR slow–down effect’ survives as the Kerr black hole is immersed into a Λ fluid which is described by the Kerr–de Sitter or Kerr–anti–de Sitter solution. This is true for Keplerian distributions of angular momenta as well as for distributions with constant angular momenta. The parameter study reveals a trend that the slow–down effect is more pronounced for negative values of Λ in both cases, the Keplerian and $l = \text{const}$ orbiters. The minimum–maximum structure occurs for black hole spins close to 1 and fairly high Λ . This is a new GR effect in Kerr–(anti–)de Sitter spacetimes.

Acknowledgments

We thank an anonymous referee who helped us to improve the paper.

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